Desloge considers two methods of differentiating G. The first, G-A(1), uses implicit differentiation. The second, G-A(2), completes the squares before differentiating. He asserts that Kane's method and G-A(1) require the same operations and hence the same labor. This assertion is not precisely true. In practice, G-A(1) requires slightly more labor due to the "overhead" operations of literally writing down the expression for G and literally differentiating it. Also, G-A(1) is more cumbersome conceptually, since it requires the analyst to introduce the ideas of G and differentiation into his thinking. For these reasons, I recommend Kane's method over G-A(1). Why complicate things by introducing G and differentiation into one's thinking and mathematics when they serve no purpose?

Desloge asserts that there may be situations in which G-A(2) is preferable to G-A(1). Possibly, but I cannot think of any. In the simple problems noted earlier, the labor required by G-A(2) was so much greater than that of G-A(1) that I did not finish it. It is evident that in problems of realistic engineering complexity, the labor required by G-A(2) will make it virtually impractical. Therefore, G-A(2) is not a serious competitor to Kane's method.

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Comment on "Relationship Between Kane's Equation and the Gibbs-Appell Equations"

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An Engineering Note is intended to communicate new developments. The article by Desloge[1] is virtually identical to a previous article by Desloge.[2] This earlier article should have been referenced in the article under discussion here.

Desloge attempts to make several points purporting to show the superiority of Gibbs' method over Kane's method. He claims that "the Gibbs-Appell equation is more manifestly a generalized equation than Kane's equations." This is certainly not true, for Kane's method allows one to introduce generalized coordinates at the outset of any problem, just as with the Gibbs-Appell equations.

In the Note, the Gibbs function S was shown to have the following form:

$$S = \tfrac{1}{2} \sum_i \alpha_i \ddot{g}_i^2 \tag{1}$$

where the α_i are constants and the g_i are functions of the coordinates and their first and second derivatives. Desloge shows that it is never necessary to expand the squares indicated in Eq. (1), but that implicit differentiation of S can be used to form the dynamical equations. His claim is that one can construct S without carrying out the squaring operations, and then identify the α_i and g_i. His claim is certainly correct, but bears further examination. The quantities α_i and g_i can also be found by inspection from the expressions for the acceleration vectors for the various body mass centers that comprise the system. A small amount of labor can be saved by never forming Eq. (1) at all. Thus Gibbs' method can be improved by bypassing the formation of the Gibbs function. Since the steps leading to Eq. (1) were left out of the Note, the earliest point at which the α_i and g_i became "visible" was presumed to be Eq. (1), which is not true. Desloge goes on to state that there may be situations in which it is preferable to utilize a Gibbs function rather than Kane's method, but he has not offered a concrete example.

The analysis presented in the Note was concerned with the formation of nonlinear dynamical equations. Quite often, the analyst is actually interested in forming linearized equations, not by simplifying nonlinear equations but by directly forming linearized equations. In this connection, Kane's method affords the analyst a definite advantage over the use of a Gibbs function, because one is free to linearize at an earlier point in the analysis. A simple example will illustrate this point.

Point O moves on a fixed circle of radius R at a constant speed. A particle P of unit mass is connected to O by a massless rod of length l. Unit vector $\boldsymbol{b}_1$ is directed in the radial direction at O, while $\boldsymbol{b}_2$ is tangent to the circle at O. Linearized equations during which the angle q between the rod and $\boldsymbol{b}_1$ remain nearly zero are to be formed.

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We start with the position vector of P:

$$\boldsymbol{r}^P = (R + \ell\cos q)\boldsymbol{b}_1 + \ell\sin q\boldsymbol{b}_2 \tag{2}$$

We next form the velocity vector for P:

$$\boldsymbol{v}^P = -\ell\sin q(\omega + \dot{q})\boldsymbol{b}_1 + [\ell\cos q(\omega + \dot{q}) + R\omega]\boldsymbol{b}_2 \tag{3}$$

where ω is the constant angular velocity of the unit vectors $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$. Now, applying Kane's method, we find the coefficient of $\dot{q}$ in Eq. (3). This is

$$\boldsymbol{v}_{\dot{q}} = \ell\sin q\boldsymbol{b}_1 + \ell\cos q\boldsymbol{b}_2 \tag{4}$$

The analysis to this point has not introduced any linearization. It is appropriate at this point to do so. We linearize Eq. (4):

$$\boldsymbol{v}_{\dot{q}} = -\ell q\boldsymbol{b}_1 + \ell\boldsymbol{b}_2 \tag{5}$$

The velocity of P may now be linearized:

$$\boldsymbol{v}^P = -\ell q\omega\boldsymbol{b}_1 + [\ell(\omega + \dot{q}) + R\omega]\boldsymbol{b}_2 \tag{6}$$

Note that Eq. (5) cannot be obtained by finding the coefficient of $\dot{q}$ in Eq. (6). The nonlinear velocity is required to find the correct linearized partial velocity vector $\boldsymbol{v}_{\dot{q}}$.

The next step is to use the linearized velocity to find the linearized acceleration vector for P:

$$\boldsymbol{a}^P = -[2\ell\omega\dot{q} + (R + \ell)\omega^2]\boldsymbol{b}_1 + \ell[\ddot{q} - \omega^2 q]\boldsymbol{b}_2 \tag{7}$$

The final step is to form the dot product $\boldsymbol{v}_{\dot{q}} \cdot \boldsymbol{a}^P$. This yields the equation of motion

$$\ell^2\ddot{q} = 0 \tag{8}$$

Now, to apply Gibbs' method to this problem, the nonlinear acceleration vector must be formed, even to use the advantageous method of implicit differentiation. The coefficient of $\ddot{q}$ in Eq. (7) is $\ell\boldsymbol{b}_2$, which differs from the correctly linearized partial velocity given in Eq. (5). If this vector is used to carry out the dot product which led to Eq. (8), we find instead

$$\ell^2\ddot{q} = \ell^2\omega^2 = 0 \tag{9}$$

which differs from Eq. (8). Equation (8) would have resulted if we had formed the nonlinear acceleration, found the coefficient of $\ddot{q}$, and then linearized. The reader can certainly appreciate the additional labor required to carry out these steps. Thus, it appears that the connection between Kane's method and the Gibbs-Appell equation is not so simple as Desloge implied. Kane's method never required the formation of nonlinear acceleration vectors, and so represents a considerable savings in labor when one is solely interested in linearized equations.

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Comment on "Relationship Between Kane's Equation and the Gibbs-Appell Equations"

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IT is not surprising that one fundamental equation of dynamics can be derived from another. Thus, for example, Lagrange's equation and Hamilton's canonical equations can be derived from the simultaneous application of D'Alembert's principle and the principle of virtual work. Yet it does not seem inappropriate to identify Lagrange's equation and Hamilton's equations as separate entities. By Desloge's own derivation,[1] the Gibbs-Appell equation, as he calls Eq. (9) in his paper, *follows* from Kane's equation, Eq. (6) in the paper. Apparently it is Kane, and not Gibbs or Appell, who recognized the intermediate step of Eq. (6) as an entirely satisfactory termination point in deriving equations of motion (with no need to introduce the S function and proceed with the Gibbs-Appell equation).

In fact, one can show that the use of Kane's equation is operationally superior to the use of the Gibbs-Appell equation. This can be brought to light by performing the task of deriving equations linearized in certain variables. The importance and engineering significance of such linearization becomes apparent in formulating the equations of motion of flexible vehicles experiencing small elastic vibrations while undergoing large "rigid-body" motion. To clarify ideas, consider the simple example of Fig. 1.

AB and BC are two massless rigid rods of lengths R and L, respectively, hinged at B; AB is driven in a horizontal plane at the angular speed $\omega(t)$; the two rods are connected by a torsional spring of stiffness k, and P is a particle of mass m. It is desired to derive equations linear in θ. The steps involved in doing this using Kane's method and the Gibbs-Appell method are shown here, where all quantities linearized in θ are marked with a tilde:

Kane's method:

Nonlinear velocity:

$$\boldsymbol{v}^P = \omega R\boldsymbol{a}_2 + L(\omega + \dot{\theta})(-\sin\theta\boldsymbol{a}_1 + \cos\theta\boldsymbol{a}_2) \tag{1}$$

Linearized partial velocity:

$$\tilde{\boldsymbol{v}}^P_{\dot{\theta}} = -L\theta\boldsymbol{a}_1 + L\boldsymbol{a}_2 \tag{2}$$

Linearized velocity:

$$\tilde{\boldsymbol{v}}^P = \omega R\boldsymbol{a}_2 + L(\omega + \dot{\theta})(-\theta\boldsymbol{a}_1 + \boldsymbol{a}_2) \tag{3}$$

Linearized acceleration:

$$\tilde{\boldsymbol{a}}^P = \boldsymbol{a}_1[-\dot{\omega}L\theta - 2\omega L\dot{\theta} - \omega^2(R + L)] + \boldsymbol{a}_2[\dot{\omega}(R + L) + L\ddot{\theta} - \omega^2 L\theta] \tag{4}$$

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